15 Fifteenth Exercise Set

15.1 Lattice Oscillations in a Triangular Lattice

Use the Lagrangian formalism developed in the previous exercise to find the eigenfrequencies and eigenmodes associated with lattice oscillations in the plane of a two-dimensional triangular lattice (see Figure 9).

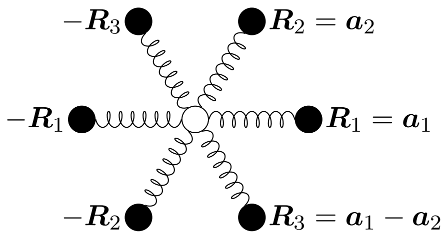

Figure 9: The nearest neighbors to the origin (center, in white) of the 2D triangular lattice used in Exercise 15.1.

For review from the previous exercise set, we had used the ansatz $\boldsymbol{u}_{\boldsymbol{R}} = \boldsymbol{u}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{R}-\omega t)}$ and left off with

$$M\omega^2 \boldsymbol{u}_0 = \left[2K \sum_{\boldsymbol{R}\in\{\mathrm{nn}(\boldsymbol{0})\}} \sin^2 \frac{\boldsymbol{k}\cdot\boldsymbol{R}}{2} (\hat{\mathbf{e}}_{\boldsymbol{R}} \otimes \hat{\mathbf{e}}_{\boldsymbol{R}})\right] \boldsymbol{u}_0.$$

We first rearrange this equation to make the underlying eigenvalue problem more clear. The result is a homogeneous system of the form

$$\left[2K \sum_{\boldsymbol{R}\in\{\mathrm{nn}(\boldsymbol{0})\}} \sin^2 \frac{\boldsymbol{k}\cdot\boldsymbol{R}}{2} (\hat{\mathbf{e}}_{\boldsymbol{R}} \otimes \hat{\mathbf{e}}_{\boldsymbol{R}}) - M\omega^2 \mathbf{I}\right] \boldsymbol{u}_0 = \mathbf{0}, \tag{15.1}$$

where $\mathbf{I}$ is the identity matrix.

For later use, we first make the auxiliary calculations shown in Table 15.1.

	$\boldsymbol{R}$	$\hat{\mathbf{e}}_{\boldsymbol{R}}$	$\hat{\mathbf{e}}_{\boldsymbol{R}} \otimes \hat{\mathbf{e}}_{\boldsymbol{R}}$	$\sin^2 \frac{\boldsymbol{k}\cdot\boldsymbol{R}}{2}$
$\pm\boldsymbol{R}_1$	$a(1,0)$	$(1,\ 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\sin^2 \frac{k_x a}{2}$
$\pm\boldsymbol{R}_2$	$a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix}$	$\sin^2 \frac{(k_x+\sqrt{3}k_y)a}{4}$
$\pm\boldsymbol{R}_3$	$a\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}$	$\sin^2 \frac{(k_x-\sqrt{3}k_y)a}{4}$

To find the lattice's eigenfrequencies, and to ensure that the eigenvalue problem in Equation 15.1 has a non-trivial solution, we require the determinant of the matrix in square brackets equal zero, i.e.

$$\det \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = 0,$$