4.3 Parity

Discuss the quantum-mechanical parity operator and give a physical interpretation of parity transformation. State and derive some of the parity operator's important quantities. Discuss the relationship of the parity operator to problems with even potentials.

- Parity transformation corresponds to space inversion, and is encoded by the parity operator $\mathcal{P}$, which maps $\boldsymbol{r}$ to $-\boldsymbol{r}$ in the form $\mathcal{P} : \psi(\boldsymbol{r}) \mapsto \psi(-\boldsymbol{r})$.
- The parity operator is Hermitian, which we prove with

$$\langle \phi(\boldsymbol{r}) | \mathcal{P} | \psi(\boldsymbol{r}) \rangle = \langle \phi(\boldsymbol{r}) | \psi(-\boldsymbol{r}) \rangle = \langle \phi(-\boldsymbol{r}) | \psi(\boldsymbol{r}) \rangle = \langle \mathcal{P}\phi(\boldsymbol{r}) | \psi(\boldsymbol{r}) \rangle .$$

  The parity operator is also unitary, i.e. $\mathcal{P}\mathcal{P} = \mathrm{I} \implies \mathcal{P} = \mathcal{P}^{-1}$.
- The parity operator changes the sign of the gradient (or derivative) operator, i.e.

$$\mathcal{P}\nabla\psi = -\nabla\mathcal{P}\psi \implies \mathcal{P}\nabla = -\nabla\mathcal{P}.$$

  The relationship $\mathcal{P}\nabla = -\nabla\mathcal{P}$ implies

$$\mathcal{P}\nabla^n = (-1)^n \nabla^n \mathcal{P} \qquad \text{and} \qquad \mathcal{P}\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{P},$$

  and the last two identities lead to

$$\mathcal{P}\boldsymbol{p} = -\boldsymbol{p}\mathcal{P} \qquad \text{and} \qquad \mathcal{P}(\boldsymbol{r} \times \boldsymbol{p}) = \mathcal{P}\boldsymbol{L} = \boldsymbol{L}\mathcal{P}.$$

- For problems with an even potential, is always possible to create an even or odd stationary state eigenfunction for each energy eigenvalue E.

Derivation: Parity Operator and an Even Potential

- For an even potential, i.e. $V(\boldsymbol{r}) = V(-\boldsymbol{r})$, the parity operator acts on V according to $\mathcal{P}V(\boldsymbol{r}) = V(-\boldsymbol{r})\mathcal{P} = V(\boldsymbol{r})\mathcal{P}$, in which case $\mathcal{P}$ and H commute, which follows from

$$\mathcal{P}H\psi(\boldsymbol{r}) = H\mathcal{P}\psi(\boldsymbol{r}) \implies [\mathcal{P}, H] = 0.$$

  If $\mathcal{P}$ and H commute, and if $|\psi(\boldsymbol{r})\rangle$ is a stationary state of the Hamiltonian and obeys the stationary Schrödinger equation

$$H\,|\psi(\boldsymbol{r})\rangle = E\,|\psi(\boldsymbol{r})\rangle\,,$$

  then $|\psi(-\boldsymbol{r})\rangle$ is also a stationary state with the same energy E, i.e.

$$H\,|\psi(-\boldsymbol{r})\rangle = E\,|\psi(-\boldsymbol{r})\rangle$$

- We can then combine the stationary state solutions $|\psi(\boldsymbol{r})\rangle$ and $|\psi(-\boldsymbol{r})\rangle$ to create the odd and even functions $|\psi_+(\boldsymbol{r})\rangle$ and $|\psi_-(\boldsymbol{r})\rangle$ according to

$$\psi_\pm(\boldsymbol{r}) = \frac{1}{\sqrt{2}}\left(\psi(\boldsymbol{r}) \pm \psi(-\boldsymbol{r})\right).$$

  In other words, for an even potential, we can always create an even or odd stationary state eigenfunction for each energy eigenvalue E (assuming E is nondegenerate).

  Note also that both $|\psi_+(\boldsymbol{r})\rangle$ and $|\psi_-(\boldsymbol{r})\rangle$ are eigenfunctions of the parity operator with eigenvalues ± 1, i.e.

$$\mathcal{P}\psi_+(\boldsymbol{r}) = \psi_+(\boldsymbol{r}) \qquad \text{and} \qquad \mathcal{P}\psi_-(\boldsymbol{r}) = -1 \cdot \psi_-(\boldsymbol{r})$$