Interference

Both diffraction and interference are fundamentally the same phenomenon: superposition of electromagnetic waves.
Superposition of vector field $\boldsymbol{E}$ applies in general.
Superposition of scalar field $E = |\boldsymbol{E}|$ applies only for EM waves with equal polarizations.
Simplification: we consider only scalar electric field magnitude. Resulting restriction: all light in this section's analyses must have the same polarization to apply superposition principles.
Assumption: we consider only superposition of *plane* waves.
Assumption: consider EM waves only in nonmagnetic materials with $\mu = 1 \implies n = \sqrt{\varepsilon}$.

Superposition of Plane Waves

Consider the two plane waves with equal frequency ω.

$E_1 = E_{1_0} e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega t + \phi_1)}$ (assume $E_{1_0} \in \mathbb{R}$)
$E_2 = E_{2_0} e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}_2 - \omega t + \phi_2)}$ (assume $E_{2_0} \in \mathbb{R}$)
$E_1 = E_{1_0} e^{i(\Phi_1 - \omega t)}$ (alternate expression; $\Phi_1 \equiv \boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \phi_1$)
$E_2 = E_{2_0} e^{i(\Phi_2 - \omega t)}$ (alternate expression; $\Phi_2 \equiv \boldsymbol{k}_2\cdot\boldsymbol{r}_2 - \phi_2$)
$E = E_{1_0} e^{i(\Phi_1 - \omega t)} + E_{2_0} e^{i(\Phi_2 - \omega t)}$ (superposed wave)

$\langle j\rangle = \frac{1}{2}\varepsilon\varepsilon_0 c|E|^2 = \frac{1}{2}\varepsilon_0 n c_0 |E|^2$ (if $\varepsilon = n^2$)
$|E| = E_{1_0}^2 + E_{2_0}^2 + E_{1_0}E_{2_0}\left(e^{i(\Phi_1-\Phi_2)} + e^{-i(\Phi_1-\Phi_2)}\right)$
$= E_{1_0}^2 + E_{2_0}^2 + 2E_{1_0}E_{2_0}\cos\Delta\Phi$ ($\Delta\Phi \equiv \Phi_1 - \Phi_2$)
$\langle j\rangle = \frac{1}{2}\varepsilon_0 n c_0\left(E_{1_0}^2 + E_{2_0}^2 + 2E_{1_0}E_{2_0}\cos\Delta\Phi\right)$
$= \langle j_1\rangle + \langle j_2\rangle + 2\sqrt{\langle j_1\rangle\langle j_2\rangle}\cos\Delta\Phi$ (and not $\langle j_1\rangle + \langle j_2\rangle$!)
$\nu \equiv \frac{j_{\max} - j_{\min}}{j_{\max} + j_{\min}} \in (0, 1)$ (interferometric visibility)

$2\sqrt{\langle j_1\rangle\langle j_2\rangle}\cos\Delta\Phi$ is observed only if $\Delta\Phi$ is *constant*!
$\Delta\Phi = \text{constant} \implies$ light must be *coherent* to observe interference

Superposition of Equal-Amplitude Plane Waves

Situation as in Superposition of Plane Waves.
Additionally assume $E_{1_0} = E_{2_0} \equiv E_0$.

$\langle j_1\rangle = \langle j_2\rangle \equiv \langle j_0\rangle = \frac{1}{2}\varepsilon_0 n c_0 E_0^2$ (if $E_{1_0} = E_{2_0}$)
$\langle j\rangle = 2\langle j_0\rangle(1 + \cos\Delta\Phi)$ (superposed intensity)
$= 4\langle j_0\rangle\cos^2\frac{\Delta\Phi}{2}$ (superposed intensity)
$\overline{\langle j\rangle} = 4j_0\overline{\cos^2\frac{\Delta\Phi}{2}} = 2j_0$ (conservation of energy)

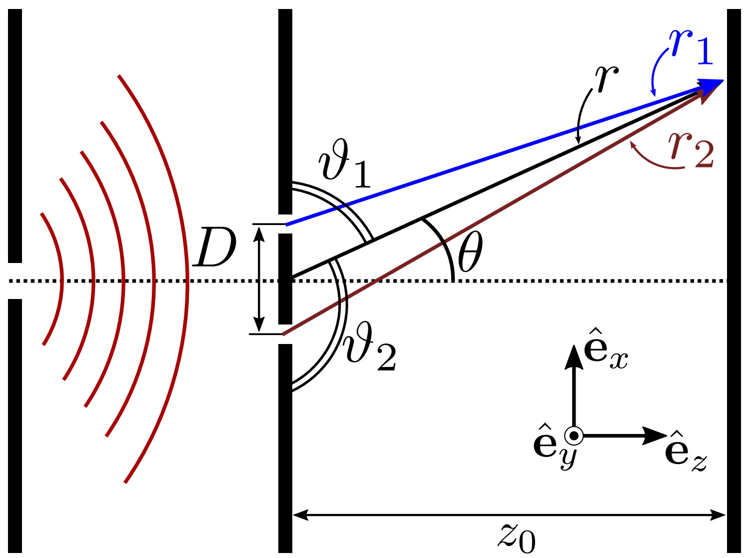

Figure 3: Geometry of Young's double slit experiment.

Young's Double-Slit Experiment

Principle: interference via wavefront splitting
Assume monochromatic point source with well-defined phase.

Young's Double-Slit Experiment

Consider two slits separated by distance D in xy plane.
z (optical) axis points from source to midpoint between slits
Let slit width run along x axis.
Work in xz plane; assume translational invariance along y axis.
Principle: thin slits split point source's spherical wavefront.
Because slits are symmetrically spaced about optical axis, light leaving each slit has equal phase.
Observe interference between light from slits on distance screen.

Geometry

r_1 and r_2 are distances from each slit to observation point.
r is distance from midpoint between slits to observation point.
θ is angle between optical axis and r.
ϑ_1 is angle between $+\hat{\mathbf{e}}_x$ and r.
ϑ_2 is angle between $-\hat{\mathbf{e}}_x$ and r.

$\vartheta_1 + \vartheta_2 = \pi \implies \cos\vartheta_2 = -\cos\vartheta_1$ (by construction)
$\vartheta_1 + \theta = \pi/2 \implies \cos\vartheta_1 = \sin\theta$ (by construction)
$r_1^2 = r^2 + (D/2)^2 - 2r(D/2)\cos\vartheta_2$ (law of cosines)
$r_2^2 = r^2 + (D/2)^2 - 2r(D/2)\cos\vartheta_1$ (law of cosines)
$r_1^2 - r_2^2 = 2rD\cos\vartheta_1$ ($\cos\vartheta_2 = -\cos\vartheta_1$)
$= 2rD\sin\theta$ ($\cos\vartheta_1 = \sin\theta$)

Assume $r_1, r_2 \gg D \implies r_1 + r_2 \approx 2r$.
$2rD\sin\theta = (r_1 + r_2)(r_1 - r_2) \approx 2r\Delta r$
$\Delta r \approx D\sin\theta$ (difference in optical path lengths to OP)

Intensity

$\Delta\Phi = k(r_1 - r_2) \approx kD\sin\theta$ (phase difference at OP)
For shorthand let $j_0 \equiv \langle j_0\rangle$.
$\langle j\rangle = 4j_0\cos^2\frac{\Delta\Phi}{2}$ (superposed intensity at OP)
$= 4j_0\cos^2\left(\frac{kD\sin\theta}{2}\right)$ (superposed intensity at OP)
$\frac{kD\sin\theta}{2} = \frac{k\Delta r}{2} = m\pi$ (condition for intensity maxima)
$r_1 - r_2 = m\lambda$ (path difference for maxima)
$\frac{kD\sin\theta}{2} = \frac{k\Delta r}{2} = \frac{\pi}{2} + m\pi$ (condition for intensity minima)
$r_1 - r_2 = \lambda(m + 1/2)$ (path difference for minima)
$\Delta\theta \approx \frac{\lambda}{D}$ (approx. angular spacing btwn. extrema)
$\Delta x \approx \frac{\lambda z_0}{D}$ (approx. position spacing btwn. extrema)

Relationship to Two-Slit Fraunhofer Diffraction

$j(\theta) = j_0\,\mathrm{sinc}^2\left(\frac{ka\sin\theta}{2}\right)\frac{\sin^2\left(\frac{kDN\sin\theta}{2}\right)}{\sin^2\left(\frac{kD\sin\theta}{2}\right)}$ (N slits of width a)
$j(\theta) \approx j_0\frac{\sin^2\left(\frac{2kD\sin\theta}{2}\right)}{\sin^2\left(\frac{kD\sin\theta}{2}\right)}$ (for $\theta \ll 1$ and two slits)
$= j_0\frac{4\sin^2\frac{kD\sin\theta}{2}\cos^2\frac{kD\sin\theta}{2}}{\sin^2\frac{kD\sin\theta}{2}}$ (trig. identities)
$= 4j_0\cos^2\left(\frac{kD\sin\theta}{2}\right)$ (same as in Young's experiment!)

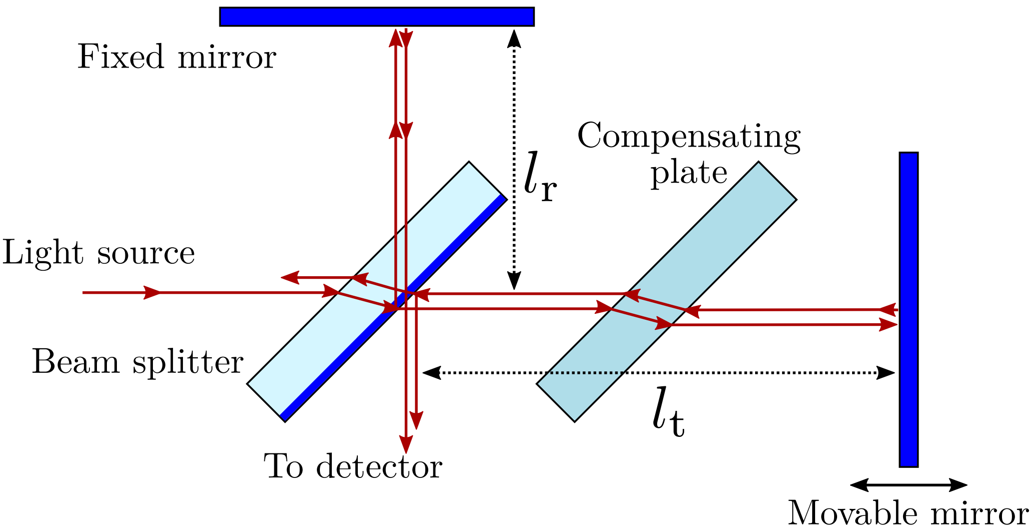

Figure 4: Geometry of a Michelson interferometer.

Interference via Amplitude Splitting

Principle: split a single incident beam into two equal parts with beam splitter. The split beams create an interference pattern.

Michelson Interferometer

Consider plane wave light incident on a beam splitter with $R = T = 0.5$. Transmitted light travels through compensator to movable mirror and back, then reflects to detector. Reflected light travels to fixed mirror and back, then on to detector. Compensator ensures reflected and transmitted beams travel equal optical paths.
l_t is distance btwn. beam splitter and mirror for transmitted beam.
l_r is distance btwn. beam splitter and mirror for reflected beam.
Transmitted and reflected beam interfere.
$\langle j_{\text{det}} \rangle = 4 j_0 \cos^2 \frac{\Delta\Phi}{2}$ (superposed intensity at detector)
$\Delta\Phi = 2k(l_1 - l_2) \equiv 2k\Delta l$ (phase difference between beams)
$\Delta l = \frac{m\pi}{k} = \frac{\lambda m}{2}$; $m \in \mathbb{N}$ (interference maxima condition)
Variation: *Twynman-Green interferometer*: light source is always a point source. Source light is first expanded with diverging lens, then collimated into a parallel beam incident on beam splitter.

Sagnac Interferometer

Mirrors arranged periodically around a circular loop.
Incident beam passes through beam splitter. Reflected and transmitted beams travel in opposite directions around the interferometer into detector, guided by mirrors.
In an inertial frame: transmitted and reflected beams travel equal optical paths. No phase difference and perfect constructive interference at detector.
Rotating frame: beams travels different optical path lengths around interferometer. Beams have difference phase at detector $\implies$ some destructive interference and weaker signal.

Sagnac Interferometer: Analysis

R is interferometer radius.
Ω is angular speed of interferometer rotation relative to inertial reference frame. Typically $\omega R \sim 1\,\mathrm{m\,s^{-1}}$.
ω is angular frequency of light waves.
t_1 is time required for beam traveling *opposite* direction of interferometer rotation to circumvent interferometer.
l_1 is orbital distance traced out by intf. edge in time t_1.
$t_1 = \frac{2\pi R - l_1(\Omega)}{c}$
$l_1(\Omega) = \Omega R t_1$
$t_1 = \frac{2\pi R}{c + \Omega R}$ (using $l_1 = \Omega R t_1$)
t_2 is time required for beam traveling *in* direction of interferometer rotation to circumvent interferometer.
l_2 is orbital distance traced out by intf. edge in time t_2.
$t_2 = \frac{2\pi R + l_2(\Omega)}{c}$
$l_2 = \Omega R t_2$
$t_2 = \frac{2\pi R}{c - \Omega R}$ (using $l_2 = \Omega R t_2$)
$\Delta t = t_2 - t_1$ (time btwn. beams reach detector)
$\Delta\Phi = \omega \Delta t$ (phase difference btwn. beams at detector)

$\Delta\Phi = \frac{4\pi R^2 \omega \Omega}{c^2 - \Omega^2 R^2} \approx \frac{4\pi S \Omega}{c^2}$ ($S = \pi R^2$; $c \gg \Omega R$)

Thin Film Interference

Consider plane waves with amplitude E_0 incident at an angle α on a thin film of width a and refractive index n_2 surrounded on either side by a material with refractive index n_1.
The incident plane wave undergoes both reflection and refraction at both film surfaces.
Goal: determine average transmitted intensity $\langle j \rangle$ on an observation screen on the opposite side of the film.
Subscript $_{12}$ denotes transition from n_1 (surroundings) to n_2 (film).
Subscript $_{21}$ denotes transition from n_2 (film) to n_1 (surroundings).

Reflection and Refraction at Boundaries

Assumption: consider only light with TE polarization
$r_{12} = \frac{n_1 \cos\alpha - n_2 \cos\beta}{n_1 \cos\alpha + n_2 \cos\beta}$ (surroundings into film)
$r_{21} = \frac{n_2 \cos\beta - n_1 \cos\alpha}{n_2 \cos\beta + n_1 \cos\alpha}$ (film into surroundings)
$r_{12} = -r_{21}$
$t_{12} = 1 + r_{12}$
$t_{21} = 1 + r_{21} = 1 - r_{12}$
$E_1 = t_{21} t_{12} E_0$ (after passing directly through film)
$E_2 = t_{21}(r_{21} r_{21} e^{i\Phi}) t_{12} E_0$ (after one internal reflection)
$E_{m+1} = t_{21} r_{21}^{2m} e^{im\Phi}) t_{12} E_0$ (after $m+1$ internal reflections)
$\Delta L = 2a\cos\beta$ (difference in optical path length between adjacent transmitted waves)
$\Phi = 2 n_2 k_0 a \cos\beta$ (phase shift btwn. adjacent trans. waves)
$\Phi = n_2 k_0 \Delta L$ (in terms of OPL difference)
$E_t = \sum_m E_m$ (total transmitted field passing through film)
$= t_{21} t_{12} E_0 \left(1 + r_{21}^2 e^{i\Phi} + r_{21}^4 e^{i2\Phi} + \cdots\right)$
$= \frac{t_{21} t_{12} E_0}{1 - r_{21}^2 e^{i\Phi}}$ (assuming $m \to \infty$)

Transmittance of Thin Films

$T = \frac{n_{\text{out}}}{n_{\text{in}}} \frac{\cos\theta_{\text{out}}}{\cos\theta_{\text{in}}} |t|^2$ (general transmittance)
$\theta_{\text{in}} = \theta_{\text{out}} = \alpha$ and $n_{\text{in}} = n_{\text{out}} = n_1$ (for a single thin film)
$T_f = |t| = \left|\frac{E_t}{E_i}\right|^2 = \frac{1}{E_0^2} \cdot \left|\frac{t_{21} t_{12} E_0}{1 - r_{21}^2 e^{i\Phi}}\right|^2$ (thin film's transmittance)
$= \frac{(1-R)^2}{1 + R^2 - 2R\cos\Phi}$ ($R \equiv |r_{12}|^2 = |r_{21}|^2$)
$= \frac{1}{1 + \frac{4R^2}{(1-R)^2} \sin^2(\Phi/2)}$ (using $\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2}$)
$\equiv \frac{1}{1 + F\sin^2(\Phi/2)}$ ($F \equiv \frac{4R}{(1-R)^2}$ is film's finesse coefficient)