Elastomechanics

A Few Vector Calculus Identities

$\phi : \mathbb{R}^3 \to \mathbb{R}$ denotes a well-behaved scalar field.
$\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a well-behaved vector field.
$v \equiv |\boldsymbol{v}|$ (shorthand for vector magnitude)
$\nabla \times (\nabla \phi) = \mathbf{0}$ (curl of gradient is zero)
$\frac{1}{2}\nabla v^2 = \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$
$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$

Common Vector Operators By Components

$(\nabla\phi)_i = \frac{\partial \phi}{\partial x_i}$ (gradient of a scalar)
$(\nabla \boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j}$ (gradient of a vector)
$\nabla \cdot \boldsymbol{v} = \frac{\partial v_i}{\partial x_i}$ (divergence of a vector)
$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i^2}$ (Laplacian of a scalar)
$(\nabla^2 \boldsymbol{v})_i = \frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_j}$ (Laplacian of a vector)
$\big[\nabla(\nabla\cdot\boldsymbol{v})\big]_i = \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$ (gradient of divergence)
$\big[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big]_i = v_j \frac{\partial v_i}{\partial x_j}$ (convective derivative)

Geometry of Deformations

Displacement Vector

Consider a reference element in a continuous body at the position vector $\boldsymbol{x}$. Due to a deformation, the reference element shifts to the new position $\boldsymbol{x}'$.
$\boldsymbol{u} \equiv \boldsymbol{x}' - \boldsymbol{x}$ (displacement vector)
Lesson: $\boldsymbol{x}'$, and thus $\boldsymbol{u}$, are functions of the initial position $\boldsymbol{x}$.

Separation Between Neighboring Elements

Consider two neighboring reference elements initially connected by the position vector $\boldsymbol{x}$. After a deformation, the elements are connected by a new position vector $\boldsymbol{x}'$.
$(\mathrm{d}l)^2 = (\mathrm{d}x_i)^2$ (pre-deformation separation btwn. elements)
$\quad = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2$ (written in full)
$\mathrm{d}\boldsymbol{u} = \mathrm{d}\boldsymbol{x}' - \mathrm{d}\boldsymbol{x}$ (displacement vector)
$\mathrm{d}\boldsymbol{x}' = \mathrm{d}\boldsymbol{x} + \mathrm{d}\boldsymbol{u}$ (new position in terms of $\boldsymbol{u}$)
$(\mathrm{d}l')^2 = (\mathrm{d}x_i')^2$ (post-deformation separation btwn. elements)
$\quad = (\mathrm{d}x_i + \mathrm{d}u_i)^2$

Deriving the Strain Tensor

$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j$ ($\mathrm{d}u$ in terms of $\mathrm{d}x$)
Substitute this expression for $\mathrm{d}u_i$ into $(\mathrm{d}l')^2$ above to get...

$$(\mathrm{d}l')^2 = \left(\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\right)^2 = (\mathrm{d}x_i)^2 + 2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k$$

$(\mathrm{d}l')^2 - (\mathrm{d}l)^2 = 2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k$ (rearranged)
$2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathrm{d}x_i\,\mathrm{d}x_j$ (symmetrization)
$\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k = \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j$ (changed dummy indices)

$$\implies (\mathrm{d}l')^2 - (\mathrm{d}l)^2 = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)\mathrm{d}x_i\,\mathrm{d}x_j$$

The above expression motivates the definition of the strain tensor as...
$u_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ (**strain tensor**)
$(\mathrm{d}l')^2 - (\mathrm{d}l)^2 = 2u_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$ (in terms of strain tensor)
$u_{ij}^{\mathrm{lin}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (linear strain tensor)

Strain Tensor and Displacement Vector's Gradient

$u_{ij}^{\mathrm{lin}} = \frac{1}{2}\big[(\nabla\boldsymbol{u})_{ij} + (\nabla\boldsymbol{u})_{ji}\big]$ (linear ST in terms of $\nabla\boldsymbol{u}$)
$\mathsf{u}_{\mathrm{lin}} = \frac{1}{2}\big[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\big]$ (in vector notation)

$$(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric}} + \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{asymmetric}}$$

$\nabla\boldsymbol{u}$'s symmetric component is the linear strain tensor.
$\nabla\boldsymbol{u}$'s asymmetric component corresponds to rigid rotations.

Strain Tensor's Symmetry and Rigid Rotations

Strain tensor is made symmetric on physical grounds so that $u_{ij} = 0$ (no internal deformation) for rigid rotations.
Consider a rigid rotation about the z axis by $\delta\phi \ll 1$.

$$\mathbf{R} = \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(rotation matrix)}$$

$$\mathbf{R} \approx \begin{pmatrix} 1 & -\delta\phi & 0 \\ \delta\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (\text{for } \delta\phi \ll 1)$$

An initial position $\boldsymbol{x} = (x_1, x_2, x_3)^\top$ transforms as...

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_2\delta\phi \\ x_1\delta\phi \\ 0 \end{pmatrix} \equiv \boldsymbol{x} + \boldsymbol{u}$$

Idea: by definition, rigid rotations don't deform bodies ($u_{ij} \equiv 0$)!
Because the strain tensor is symmetrized...
$u_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\delta\phi + \delta\phi = 0$ (correctly, $u_{ij} = 0$)
$u_{21} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \delta\phi - \delta\phi = u_{12} = 0$ (correctly, $u_{ij} = 0$)
If the strain tensor were not symmetrized...
$\widetilde{u}_{12} = \frac{\partial u_1}{\partial x_2} = -\delta\phi$ (non-physically, $\widetilde{u}_{12} \neq 0$)
$\widetilde{u}_{21} = \frac{\partial u_2}{\partial x_1} = \delta\phi$ (non-physically, $\widetilde{u}_{12} \neq 0$)

Physical Meaning of the Diagonal Components

No summation implied over α!
Diagonal components $u_{\alpha\alpha}$ encode extensional strains along the α coordinate axes, e.g. u_{xx} is extensional strain along the x axis.
Consider two neighboring body elements with reference separation Δl. A deformation then separates the elements to...
$(\Delta l')^2 = (\Delta l)^2 + 2u_{ik}\Delta x_i \Delta x_k$
Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ denote the strain tensor's principal axes.
$\Delta l' = \sqrt{1 + 2u_{\alpha\alpha}}\Delta l$ (elements with ref. spacing $\Delta\boldsymbol{x} = \Delta l\,\hat{\mathbf{e}}_1$)
$\Delta l' \approx (1 + u_{\alpha\alpha})\Delta l$ (for small strains $u_{\alpha\alpha}$)
$u_{\alpha\alpha} = \frac{\Delta l' - \Delta l}{\Delta l}$ (diag. components are extensional strains)

Physical Meaning of the Off-Diagonal Components

Consider two pairs of nearby body elements with separations...
$\Delta\boldsymbol{x}_1 = \Delta x_1\,\hat{\mathbf{e}}_1$ and $\Delta\boldsymbol{x}_2 = \Delta x_2\,\hat{\mathbf{e}}_2$
A deformation then transforms the separations to...

$$\Delta\boldsymbol{x}_1' = \Delta\boldsymbol{x}_1 + \frac{\partial\boldsymbol{u}}{\partial x_1}\Delta x_1 = \left[\left(1 + \frac{\partial u_1}{\partial x_1}\right)\hat{\mathbf{e}}_1 + \frac{\partial u_2}{\partial x_1}\hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial x_1}\hat{\mathbf{e}}_3\right]\Delta x_1$$

$$\Delta\boldsymbol{x}_2' = \Delta\boldsymbol{x}_2 + \frac{\partial\boldsymbol{u}}{\partial x_2}\Delta x_2 = \left[\frac{\partial u_1}{\partial x_2}\hat{\mathbf{e}}_1 + \left(1 + \frac{\partial u_2}{\partial x_2}\right)\hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial x_2}\hat{\mathbf{e}}_3\right]\Delta x_2$$

To lowest order in products of Δx_1, Δx_2 and $\frac{\partial u_i}{\partial x_j}$...

$$\Delta\boldsymbol{x}_1' \cdot \Delta\boldsymbol{x}_2' \approx \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\Delta x_1 \Delta x_2 = 2u_{12}\Delta x_1 \Delta x_2.$$

$\cos\theta_{12} = \frac{\Delta\boldsymbol{x}_1' \cdot \Delta\boldsymbol{x}_2'}{|\Delta\boldsymbol{x}_1'||\Delta\boldsymbol{x}_2'|} \approx 2u_{12}$ (angle between $\Delta\boldsymbol{x}_1'$ and $\Delta\boldsymbol{x}_2'$)
$\theta_{12} \approx \frac{\pi}{2} - 2u_{12}$ (using $\arccos x = \frac{\pi}{2} - x + \mathcal{O}(x^3)$)
$\Delta\theta_{12} \equiv \frac{\pi}{2} - \theta_{12} = 2u_{12}$
$u_{\alpha\beta} = \frac{1}{2}\Delta\theta_{\alpha\beta}$ (meaning of off-diagonal components)
$\Delta\theta_{\alpha\beta}$ is the post-deformation reduction in angle (from the initially perpendicular value $\pi/2$) between a pair of line elements $\Delta\boldsymbol{x}_\alpha$ and $\Delta\boldsymbol{x}_\beta$ initially parallel to $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$, respectively.

Relative Change in Volume

Consider a cuboid reference body element with volume ΔV.
A deformation transforms the element to have volume $\Delta V'$.
$x_i' = x_i + u_i(x_j)$ (post-deformation coordinates)
$\mathbf{J}_{ij} \equiv \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$ (Jacobian matrix)

$$\mathbf{J} = \begin{pmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$\Delta V' \approx \det\mathbf{J} \cdot \Delta V$ (to lowest order in Δx_j)
$\det\mathbf{J} \approx 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \operatorname{tr}\mathsf{u}$ (to first order in $\frac{\partial u_i}{\partial x_j}$)
$\operatorname{tr}\mathsf{u} = \frac{\Delta V' - \Delta V}{\Delta V}$ ($\operatorname{tr}\mathsf{u}$ gives relative change in volume)